

About Hybrid System Stability with Distributed and Concentrated Parameters

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Abstract

The paper deals with the problem of hybrid system stability with distributed and concentrated parameters, described by interrelated linear differential equations in partial and ordinary derivatives. The problem is solved by the method of Lyapunov's functions (functionals). The main problem in the use of the method is the development of the corresponding Lyapunov's functions (functionals), which were usually constructed intuitively, starting from the total energy, first integrals and other considerations during the study of specific objects with distributed parameters. In order to facilitate the solution of this problem, it is proposed to transform first of all the initial differential equations of high order into the system of equations of the first order. During the decrease of equation order in the private derivatives of the higher order by the introduction of additional variables for the derivatives according to the spatial coordinates, the equations without time derivatives appear. The technique for stability study has been developed in the work for these systems, some of equations of which do not contain time derivatives. The developed technique, related with the idea of partial differential equation transformation of high order into the system of equations of the first order both in time and in spatial coordinates, facilitates a lot the construction of Lyapunov's functionals in the form of integral quadratic forms. The construction of such forms is carried out constructively on specific equations. The obtained results allow to study the stability of a wide class of engineering objects with distributed and concentrated parameters. The stability of a rotor-type wind engine with a working machine (generator, pump, compressor, etc.) considering the elasticity of the shaft transferring the torque from the wind engine to the machine is considered as an illustrative example. The use of environmentally friendly wind engines, which allow to reduce energy costs, is a promising area. A wind engine of a rotor type is silent, safe and, unlike propeller wind engines, can be placed near settlements and buildings.

Keywords: Stability, Hybrid Systems with Distributed and Concentrated Parameters, The Method of Lyapunov's Functions.

Introduction

One of the main methods for the study of system stability with distributed parameters is the method of Lyapunov functions (functionals). Besides, the hybrid systems with distributed and concentrated parameters

were also considered along with the purely distributed ones. A fairly complete survey in this area can be found in [1 - 3]. Along with theoretical studies, the method of Lyapunov's functions was used to study the specific objects with distributed parameters. For example, these objects are represented by elastic and aeroelastic objects [4 ± 7], chemical reactors [1], liquid rocket engines [1, 4], magneto dynamic processes [1], etc. [1, 4]. In the applications, the main problem is the development of the corresponding Lyapunov's functionals, which were usually constructed intuitively, starting from the total energy, first integrals, and other considerations. In [8], a new approach was proposed during the solution of problems concerning the stability systems with distributed parameters, the essence of which is the following one. The equations of higher-order private derivatives are first reduced to the system of equations of the first order in time and spatial coordinates. Further, the Lyapunov's functionals are developed for this system with respect to concrete equations in the form of integral quadratic forms whose definite signs can be verified by the application of the well-known Sylvester criterion. This approach allows us to design the Lyapunov's functionals constructively and extends the use of the Lyapunov's function method in specific applications greatly. Using it, for example, rather complex problems of flexural-torsional oscillations stability are solved concerning an elastic wing of an aircraft [4], the systems with distributed parameters and concentrated forces [9], etc. [4].

In this paper, the approach described above is applied to the study of hybrid system stability with distributed and concentrated parameters. For some hybrid systems, this approach was used earlier [4, 10] during the study of stability with respect to some variables.

Methods

The work uses universally recognized, mathematically rigorous and precise methods of research. The main

one is the method of Lyapunov's functions, which, was created originally for finite-dimensional systems and turned out to be more suitable for distributed and hybrid systems, since the use of other methods is difficult here.

During the calculation of the derivative from the Lyapunov's function (2.1), by virtue of equations (1.1) - (1.3), the approach is used to take into account the equations without derivatives, which by its procedure is similar to the method of Lagrange multipliers in variational problems.

The stability conditions are written on the basis of well-known classical results on the stability of finite-dimensional and distributed systems. The check of square form sign determination is carried out by the Sylvester's criterion.

Results and Discussion

1. Statement of the problem. Let's consider a hybrid system with one distributed and other finite-dimensional links, the perturbed state of which is described by the following equations

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= A(x) \frac{\partial \varphi}{\partial x} + B(x) \frac{\partial \psi}{\partial x} + A_0(x)\varphi + B_0(x)\psi, \\ C(x) \frac{\partial \varphi}{\partial x} + D(x) \frac{\partial \psi}{\partial x} + C_0(x)\varphi + D_0(x)\psi, \\ x &\in (0,1), \\ \frac{dz}{dt} &= F_1 z + F_2 \varphi(0,t) + F_3 \varphi(1,t) + F_4 \psi(0,t) + F_5 \psi(1,t), \\ (1.2) \quad \Gamma_1 \varphi(0,t) &= \Gamma_2 z, \quad \Gamma_3 \varphi(1,t) = \Gamma_4 z, \\ t &\in I = (0, \infty), \end{aligned}$$

where $\varphi = \varphi(x, t)$ is the vector of phase functions of the distributed link, $\psi = \psi(x, t)$ – is the vector of phase functions of this link whose time derivatives do not enter the system (1.1), $z = z(t)$ – the vector of phase functions of finite-dimensional links, $A(x)$, $B(x)$, $C(x)$, $D(x)$, $A_0(x)$, $B_0(x)$, $C_0(x)$, $D_0(x)$ – the matrices whose elements are bounded by continuous functions, $F_1, \dots, F_5, \Gamma_1, \dots, \Gamma_4$ – the matrices of constants.

From the mathematical point of view, the problem (1.1) - (1.3) is a boundary value problem for private derivative equations under complicated boundary conditions. The equations (1.1) represent the

general record form of any linear equation of arbitrary order private derivatives in the form of a system of partial differential equations of the first order [4, 8]. In order to transform the higher-order equations to the form (1.1), we must take the lower derivatives for additional variables and put down the initial equation and the integrability condition in these variables.

The equations (1.3) are simple boundary conditions connecting the boundary values of the components $\varphi(x, t)$ with each other or with the variable z (see the third and the fourth equations (3.1) in the example). The equation of dynamics (1.2) for finite-dimensional links located at both ends of a distributed link contains boundary values $\varphi(x, t)$ and represents a complicated boundary condition in the form of a differential equation (the first equation (3.1)).

The systems with the transmission shafts of considerable length, for example, between an engine and a working machine (a generator, a pump, a compressor, etc.); the systems containing electrical circuits or pipelines (trunk lines) in which it is necessary to take into account the distributed nature of liquid or gas flow, etc. are described by the equations of the type (1.1) - (1.3)

Let's introduce the following measure

$$\rho = \int_0^1 \varphi^T \varphi dx,$$

which characterizes the disturbed state of the distributed link, and consider the asymptotic stability of the solution $\varphi = \psi = z = 0$ for the system (1.1) – (1.3) according to the variables z, ρ . (1.3)

1. Stability study. In order to solve the problem, let's consider the Lyapunov's function

$$V = V_1 + V_2 = \int_0^1 \varphi^T(x, t) v(x) \varphi(x, t) dx + z^T(t) Q z(t), \quad (2.1)$$

where $v(x)$, Q are the symmetric matrices: the elements Q are constants, and the elements $v(x)$ are continuous bounded functions.

The peculiarity of the system (1.1) - (1.3) is that the second equation (1.1) and the equation (1.3) do not contain time derivatives t . This does not allow us to calculate the derivative V directly by the virtue of the entire system.

Here we use the approach given earlier [8], and first

let's compute the derivative dV/dt by the virtue of the first equation (1.1) and the equation (1.2):

$$\begin{aligned} \frac{dV}{dt} = & \int_0^1 \left[\varphi^T v \left(A \frac{\partial \varphi}{\partial x} + B \frac{\partial \psi}{\partial x} \right) + \left(\frac{\partial \varphi^T}{\partial x} A^T + \frac{\partial \psi^T}{\partial x} B^T \right) v \varphi + \right. \\ & + \varphi^T \left(v A_0 + A_0^T v \right) \varphi + \varphi^T v B_0 \psi + \psi^T B_0^T v \varphi \left. \right] dx + \\ & + z^T \left(Q F_1 + F_1^T Q \right) z + 2 \varphi^T(0, t) F_2^T Q z + 2 \varphi^T(1, t) F_3^T Q z + \\ & + 2 \psi^T(0, t) F_4^T Q z + 2 \psi^T(1, t) F_5^T Q z. \end{aligned}$$

Following the method of Lagrange multipliers, in order to take into account the second equation (1.1) and the equations (1.3), let's add the following equalities to this derivative:

$$\begin{aligned} & \int_0^1 \left[\left(\varphi^T P_1 + \psi^T P_2 \right) \left(C \frac{\partial \varphi}{\partial x} + D \frac{\partial \psi}{\partial x} + C_0 \varphi + D_0 \psi \right) + \right. \\ & \left. + \left(\frac{\partial \varphi^T}{\partial x} C^T + \frac{\partial \psi^T}{\partial x} D^T + \varphi^T C_0^T + \psi^T D_0^T \right) \left(P_1^T \varphi + P_2^T \psi \right) \right] dx = 0, \end{aligned}$$

$$\begin{aligned} & \left[\varphi^T(0, t) R_1 + z^T R_2 \right] \left[\Gamma_1 \varphi(0, t) - \Gamma_2 z \right] + \left[\varphi^T(0, t) \Gamma_1^T - z^T \Gamma_2^T \right] \times \\ & \times \left[R_1^T \varphi(0, t) + R_2^T z \right] = 0, \end{aligned}$$

$$\begin{aligned} & \left[\varphi^T(1, t) R_3 + z^T R_4 \right] \left[\Gamma_3 \varphi(1, t) - \Gamma_4 z \right] + \left[\varphi^T(1, t) \Gamma_3^T - z^T \Gamma_4^T \right] \times \\ & \times \left[R_3^T \varphi(1, t) + R_4^T z \right] = 0, \end{aligned}$$

where $P_1 = P_1(x)$, $P_2 = P_2(x)$, R_1, \dots, R_4 – while arbitrary matrices P_1, P_2 are continuous ones, R_1, \dots, R_4 are the constant ones. The

brackets $\left(\varphi^T P_1 + \psi^T P_2 \right)$, $\left(P_1^T \varphi + P_2^T \psi \right)$, $\left[\varphi^T(0, t) R_1 + z^T R_2 \right]$, $\left[R_1^T \varphi(0, t) + R_2^T z \right]$ play the role of Lagrange multipliers.

Let's perform the integration by parts and require that the matrices $v, P_1, P_2, Q, R_1, \dots, R_4$ satisfy the following equations:

$$\begin{aligned} vA + P_1 C &= A^T v + C^T P_1^T, \quad P_2 D = D^T P_2^T, \\ vB + P_1 D &= C^T P_2^T, \quad P_2 D_0 + D_0^T P_2^T = \frac{dP_2 D}{dx}, \\ vB_0 + P_1 D_0 + C_0^T P_2^T &= \frac{d(vB + P_1 D)}{dx}, \quad x \in (0,1). \end{aligned}$$

And boundary conditions at $x = 0$ and $x = 1$:

$$\begin{aligned} v(0)A(0) + P_1(0)C(0) - R_1 \Gamma_1 - \Gamma_1^T R_1^T &= 0, \\ QF_2 + R_2 \Gamma_1 - \Gamma_2^T R_1^T &= 0, \\ v(1)A(1) + P_1(1)C(1) + R_3 \Gamma_3 + \Gamma_3^T R_3^T &= 0, \\ QF_3 + R_4 \Gamma_3 - \Gamma_4^T R_3^T &= 0, \\ (P_2 D) \Big|_0^1 = (vB + P_1 D) \Big|_0^1 = QF_4 = QF_5 &= 0. \end{aligned}$$

Then we obtain the following equation for the derivative dV/dt due to the system (1.1) – (1.3):

$$\frac{dV}{dt} = - \int_0^1 \varphi^T w \varphi dx - z^T H z,$$

i.e. the square form of the same type as for V (2.1). Here

$$w = \frac{d(vA + P_1 C)}{dx} - vA_0 - A_0^T v - P_1 C_0 - C_0^T P_1^T,$$

$$H = - \left(QF_1 + F_1^T Q \right) + R_2 \Gamma_2 + \Gamma_2^T R_2^T + R_4 \Gamma_4 + \Gamma_4^T R_4^T. \tag{2.6}$$

The obtained results enable us to solve the problem of the quadratic form V (2.1) development. To do this, we must specify the symmetric matrix $w(x)$ and solve the equations (2.2), (2.5) with respect to the matrices v, P_1, P_2 under the boundary conditions following from the equations (2.3). However, in contrast to the problem of quadratic form development in the case of linear ordinary differential equations, not

all elements of the matrix w can be defined arbitrarily here, some of them are determined in the course of the problem solution (see the example). From equations (2.3) we also find the matrices Q, R_1, R_2, R_3, R_4 .

According to the method of Lyapunov's functions [1, 4], the zero solution of (1.1) - (1.3) system will be asymptotically stable with respect to the variables z, ρ , if the integral quadratic form V_1 (2.1) is continuous and definitely positive in measure ρ , the quadratic form V_2 (2.1) is definitely positive, and the derivative dV/dt (2.4) is definitely negative in terms of the variables z, ρ .

The continuity of the integral form V_1 with respect to the measure ρ follows directly from the continuity, and, consequently, the boundedness of the matrix $v(x)$ elements at $x \in [0,1]$. The rest of the conditions of this statement will be satisfied if Q, H are definitely positive, and the matrices $v(x), w(x)$ are definitely positive at $x \in [0,1]$, i.e.

$$Q > 0, H > 0; v(x) > 0, w(x) > 0, x \in [0,1].$$

- Example.** Let us consider the stability of a rotor-type wind engine with a vertical axis of rotation [10, 11] together with a working machine (a generator, a pump, etc.). The shaft, which transmits the torque of the wind engine to the machine, has a considerable length, so the problem is solved taking into account the elasticity of this shaft.

Let's note that the use of environmentally friendly wind engines, which allow to reduce energy costs, is a promising trend. The rotor-type wind engine does not require a system of orientation to the wind, it is noiseless, safe and, unlike propeller wind engines, can be placed near settlements and buildings.

The equations of an engine dynamics [12] with a working machine and with an elastic transfer shaft will be recorded with additional consideration of the elasticity of the end of the transmission shaft insertion into the machine shaft in relative deviations from a nominal operating mode of the wind engine with a machine

$$\frac{dz}{dt} = kz + \frac{\partial \varphi(x,t)}{\partial x} \Big|_{x=0},$$

$$\frac{\partial^2 \varphi(x,t)}{\partial t^2} = a \frac{\partial^2 \varphi(x,t)}{\partial x^2}, \quad x \in (0,1),$$

$$\frac{\partial \varphi(x,t)}{\partial x} \Big|_{x=1} = -k_1 \varphi(1,t) - k_2 \frac{\partial \varphi(x,t)}{\partial t} \Big|_{x=1}, \quad \frac{\partial \varphi(x,t)}{\partial t} \Big|_{x=0} = k_3 z.$$

Here $x = \frac{y}{\ell}, \quad z = \frac{\omega - \omega_*}{\omega_*},$

$$\varphi(x,t) = \frac{\psi(x,t) - \psi_*(x,t)}{\psi_{\max*}},$$

$$\psi_{\max*} = \frac{M_* \ell}{GI}, \quad a = \frac{GI}{J \ell^2},$$

$$k = \frac{1}{J_b} \left(\frac{\partial M_b}{\partial \omega} \right)_*, \quad k_1 = \frac{\lambda}{GI},$$

$$k_2 = \frac{\ell}{GI} \left(\frac{\partial M}{\partial \omega} \right)_*, \quad k_3 = \frac{GI \omega_*}{\ell M_*}, \quad y - \text{are the}$$

coordinates of the transmission shaft cross sections, ω – the angular velocity of a wind engine, $\psi(x,t), \ell, J, GI$ – an absolute angle of section rotation, length, running moment of inertia, torsional stiffness of the transmission shaft, M_b, M – the torque of a wind engine and a machine, J_b – the moment of a wind engine inertia, $\psi_{\max*}$ – the elasticity factor of the end of the transmission shaft in the machine shaft, λ – the elasticity factor of the end of the transmission shaft insertion in the machine shaft, the sign (*) indicates the values of the values in the rated operating mode of a wind engine with the machine, when $\omega_* = const,$

$$M_b = M = M_* = const.$$

Introducing new variables $\varphi_1 = \varphi(x,t), \varphi_2 = \partial \varphi / \partial t, \varphi_3 = \partial \varphi / \partial x$ and taking into account the integrability condition $\partial \varphi_3 / \partial t = \partial \varphi_2 / \partial x$, let's put down the equations (3.1) in the form of the system (1.1) - (1.3), where

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & 1 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \|1 \ 0 \ 0\|, \quad C_0 = \|0 \ 0 \ -1\|,$$

$$F_1 = k, F_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \Gamma_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \Gamma_2 = k_3, \Gamma_3 = \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix},$$

and the matrices $B, B_0, D, D_0, F_3, F_4, F_5, \Gamma_4$ are zero ones.

Let's develop the functional (2.1), where, in this example, we take $V_2 = qz^2$, $q = const > 0$. Let's put down the equations (2.2), (2.5) in the scalar form

$$P_{21} = v_{13}, P_{31} = av_{12}, v_{33} = av_{22}.$$

$$\frac{dP_{11}}{dx} = w_{11}, \frac{dv_{23}}{dx} - 2v_{12} = w_{22},$$

$$a \frac{dv_{23}}{dx} + 2av_{12} = w_{33}, a \frac{dv_{22}}{dx} = w_{23},$$

$$a \frac{dv_{12}}{dx} + P_{11} = w_{13}, \frac{dv_{13}}{dx} - v_{11} = w_{12},$$

where P_{ij}, v_{ij}, w_{ij} are the elements of the matrices P, v, w .

The following boundary conditions are obtained from the equations (2.3) at $x = 0$ and $x = 1$:

$$P_{11}(0) = v_{23}(0) = v_{13}(0) = 0, q = ak_3v_{22}(0), \tag{3.4}$$

$$P_{11}(1) - 2ak_1v_{12}(1) - ak_1^2v_{23}(1) = 0, \tag{3.5 a)}$$

$$\left(1 + ak_2^2\right)v_{23}(1) - 2ak_2v_{22}(1) = 0, \tag{3.5 b)}$$

$$v_{13}(1) - ak_2v_{12}(1) - k_1[v_{33}(1) - ak_2v_{23}(1)] = 0.$$

Suppose that $w_{11} = const, w_{22} = 1,$

$$w_{33} = a, w_{13} = P_{11}, w_{12} = w_{23} = 0.$$

Solving first five equations (3.3) at the boundary conditions (3.4), (3.5 a), (3.5 b), we obtain the following:

$$v_{22} = \frac{(1 + ak_2^2)}{2ak_2}, v_{12} = 0, v_{23} = x, P_{11} = w_{11}x, w_{11} = P_{11}(1) = ak_1^2, w_{13} = ak_1^2x.$$

From the last equations (3.2), (3.4) it follows

$$\text{that } v_{33} = av_{22}, q = \frac{(1 + ak_2^2)k_3}{2k_2}, \text{ and from (2.6)}$$

we obtain $H = -2qk$.

The functions v_{11} and v_{13} occur only in the last equation (3.3). Assuming that $v_{11} = const$, we obtain the following from this equation and the boundary condition (3.5 b)

$$v_{11} = \frac{k_1(1 - ak_2^2)}{2k_2}, v_{13} = v_{11}x.$$

The functions V (2.1) and its derivative dV/dt (2.4) due to the system (3.1) will be the following ones

$$V = \int_0^1 \left[\frac{k_1(1 - ak_2^2)}{2k_2} (\varphi_1^2 + 2x\varphi_1\varphi_3) + \frac{1 + ak_2^2}{2ak_2} (\varphi_2^2 + a\varphi_3^2) + 2x\varphi_2\varphi_3 \right] dx + \frac{(1 + ak_2^2)k_3}{2k_2} z^2, \tag{3.3}$$

$$\frac{dV}{dt} = - \int_0^1 \left[ak_1^2 (\varphi_1^2 + 2x\varphi_1\varphi_3) + \varphi_2^2 + a\varphi_3^2 \right] dx + \frac{kk_3(1 + ak_2^2)}{k_2} z^2.$$

Since $k_1 > 0, k_3 > 0$, then from the inequalities (2.7) we obtain the conditions for the asymptotic stability of the wind pump assembly operation taking into account the elasticity of the transmission shaft and the expression for the coefficient k in the form

$$\left(\frac{\partial M_b}{\partial \omega} \right)_* < 0, 0 < k_2 < \frac{1}{\sqrt{a}}.$$

From a comparison of these conditions with the stability condition $\left(\frac{\partial M}{\partial \omega} \right)_* - \left(\frac{\partial M_b}{\partial \omega} \right)_* > 0$ (3.5 a)

a wind pump assembly with a rigid transfer shaft, it can be seen that the elasticity of the shaft narrows the stability area.

Conclusions

- Sufficient conditions are obtained for the stability of hybrid systems, which are represented in the form of inequalities connecting the coefficients (parameters) of the system.
- The developed methodology to study the stability of hybrid systems in comparison with known results is more constructive and

universal, i.e. allows us to study the stability of a wider class of objects with distributed and concentrated parameters.

- The work has a theoretical and practical value.

Summary

The transformation of differential equations in the private derivatives of higher order into the system of first-order partial differential equations, together with the record of ordinary differential equations in the normal form of Cauchy, allows to design constructively the Lyapunov's function as a sum of integral and ordinary quadratic forms, and to develop a general methodology of stability study for hybrid systems with distributed and concentrated parameters.

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